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ASYMPTOTIC BOUNDS FOR EXTENSION OF BONDED ELASTIC CYLINDERS

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Abstract—A right elastic cylinder has its plane ends bonded to rigid plates or supports and these undergo a relative axial translation while the curved surface is free. A Young type modulus is defined, and simple, explicit, asymptotic upper and lower bounds are obtained for this which are relevant to short and long cylinders respectively.

INTRODUCTION

A right cylinder, consisting of homogeneous isotropic elastic material, has its plane ends bonded to rigid plates or supports which undergo a relative axial translation e while its curved surface is free. If $\bar{F}$ denotes the total axial force required, and l, A denote the cylinder length and cross-sectional area respectively, the (modified Young's) modulus $\bar{E}$ is defined to be

$$\bar{E} = \bar{F}l/eA.$$

The purpose of this paper is to prove, *for a general class of cross-sections,* that

$$E(1 - \sigma)/\{(1 + \sigma)(1 - 2\sigma)\} \geq \bar{E} \geq E$$

where E, σ denote Young's modulus and Poisson's ratio respectively for the material; that these bounds are approached as $l/A^{1/2} \to 0, \infty$ respectively; and to give simple, explicit (albeit conservative) estimates of the rates at which the bounds are approached. The results are embodied in two theorems.

Various other aspects of the extension of such a bonded elastic cylinder have been considered in [1–9] but none of these deals with the actual matters dealt with in this paper.

It is assumed throughout that

$$E > 0, \qquad 0 \leq \sigma < \tfrac{1}{2}$$

although the limit of incompressibility ($\sigma \to \frac{1}{2}$) is sometimes contemplated.

1. LONG CYLINDERS

The rectangular cartesian displacement components u_i are such that

$$u_1 = u_2 = 0, \quad u_3 = \pm e/2 \quad \text{on} \quad x_3 = \pm l/2 \tag{1.1}$$

(the plane ends of the cylinder), and are such that the curved surface is free. The displacement field u_i is represented as a superposition of two equilibrium fields (without body forces):

$$u_i = u_i' + u_i'' \tag{1.2}$$

where

(i) $u_1' = -\sigma e x_1/l$, $u_2' = -\sigma e x_2/l$, $u_3' = e x_3/l$
the corresponding Saint-Venant field,

(ii) u_i'' is determined by the conditions

$$u_1'' = \sigma e x_1/l, \quad u_2'' = \sigma e x_2/l, \quad u_3'' = 0 \quad \text{on} \quad x_3 = \pm l/2$$

with free curved surface. It may be verified, by using the Reciprocal Theorem for example, that the energy ϵ associated with u_i equals the sum of the energies ϵ', ϵ'' associated with u_i', u_i'' respectively:

$$\epsilon = \epsilon' + \epsilon''. \tag{1.3}$$

It is easily verified that

$$\epsilon = \tfrac{1}{2}\bar{E}e^2A/l, \quad \epsilon' = \tfrac{1}{2}Ee^2A/l,$$

and using this, eqn (1.3) and the Principle of Minimum Potential Energy, we obtain

$$e^2A/l(\bar{E} - E) = 2\epsilon'' \leq 2\int W(e_{ij}^*)\,\mathrm{d}v \tag{1.4}$$

where $W(e_{ij}^*)$ denotes the strain energy density associated with a kinematically admissible displacement field u_i^*; such a field is furnished by

$$\begin{aligned}(\sigma e/l)^{-1}u_1^* &= x_1f(x_3)\\ (\sigma e/l)^{-1}u_2^* &= x_2f(x_3)\\ (\sigma e/l)^{-1}u_3^* &= h(x_3)\end{aligned} \tag{1.5}$$

where f, g are arbitrary, piecewise continuously differentiable functions satisfying

$$f(\pm l/2) = 1, \quad h(\pm l/2) = 0. \tag{1.6}$$

The field u_i^* is deliberately chosen so that its divergence may be zero in the right circumstances in order to cater for the case of an incompressible material.

Using (1.4) and (1.5) one obtains

$$\bar{E}/E - 1 \leq l^{-1}\sigma^2(1+\sigma)^{-1}\int\{\sigma(1-2\sigma)^{-1}(2f+h')^2 + (2f^2 + h'^2 + \tfrac{1}{4}k^2f'^2)\}\,\mathrm{d}x_3 \tag{1.7}$$

where k is the radius of gyration of the cross-section. The best bound of the type (1.7) is obtained by minimizing the integral. This is done by solving the Euler-Lagrange differential equations

$$\begin{aligned}k^2f'' - 4(1-2\sigma)^{-1}(f + \sigma h') &= 0\\ 2\sigma f' + (1-\sigma)h'' &= 0\end{aligned} \tag{1.8}$$

subject to the appropriate boundary conditions. We embody the bound so found together with the non-negativity of $\bar{E} - E$ [which follows from (1.4) and the non-negativity of strain-energy (density)] in the following theorem:

Theorem 1: *The modulus* $\bar{E}$ *satisfies*

$$1 \leq \bar{E}/E \leq \{1 - 2\sigma^2(1-\sigma^2)^{-1/2}kl^{-1}\tanh\{[(1+\sigma)/(1-\sigma)]^{1/2}lk^{-1}\}\}^{-1} \tag{1.9}$$

E, σ denoting Young's modulus and Poisson's ratio respectively, l, k denoting the length and centroidal radius of gyration of cross-section of the cylinder respectively.

As our interest lies in behaviour for large lk^{-1}, little is lost in replacing tanh by its upper bound unity in (1.9)–its asymptotic value which it approaches rapidly with increasing argument-provided denominator in resulting bounds is positive.

We note the following simple consequences of (1.9):

(a)

$$\bar{E}/E \sim 1 \quad \text{as} \quad lk^{-1} \to \infty$$

a rigorous proof of which does not appear to be available in the literature.

(b) The bound

$$\bar{E}/E \leq [1 - kl^{-1}/\sqrt{3}]^{-1} \quad \text{(provided } l > k/\sqrt{3}\text{)}$$

corresponding to an incompressible material $\sigma = \frac{1}{2}$ is valid for all values of σ.

(c) For an incompressible (or other) circular cylinder of length to diameter ratio 10:1 the percentage difference between $\bar{E}$ and E is, *at most*, just slightly in excess of 2%.

Remark

Suppose that the cylinder is compressed by the application to the cylinder ends of parallel flat rough plates which maintain contact with the cylinder ends at all points—as distinct from plates which are *perfectly* bonded to the ends—the curved surface again being free. Suppose that the plates undergo an axial translation e with respect to one another, and suppose $\tilde{F}$ is the axial force required. If the (quasi-Young's) modulus $\tilde{E}$ is defined by

$$\tilde{E} = \tilde{F}l/eA$$

then an argument analogous to that of Shield and Anderson [9] establishes that

$$\bar{E} \geq \tilde{E} \geq E.$$

In addition to the standard assumptions of elasticity theory, the proof requires the further (eminently reasonable) assumption that the end (frictional) tractions $\tilde{T}_\alpha$ and end displacements $\tilde{u}_\alpha$ satisfy

$$\tilde{T}_\alpha \tilde{u}_\alpha \leq 0 \quad (\alpha = 1, 2).$$

Thus Theorem 1 remains valid if $\bar{E}$ is replaced by $\tilde{E}$.

2. SHORT CYLINDERS

The actual displacement field u_i is represented as a superposition of two equilibrium fields (without body forces)

$$u_i = u_i^- + u_i^{**} \tag{2.1}$$

where

(i)

$$u_i^- = (ex_3/l)\delta_{i3} \tag{2.2}$$

[δ denoting the Kroenecker delta], to maintain which, traction components

$$T_\alpha^- = \lambda n_\alpha, \quad T_3^- = 0$$

($\alpha = 1, 2$) must be applied to the curved surface, n_α being the components of the unit outward normal thereto, and

$$\lambda = E\sigma/\{(1 + \sigma)(1 - 2\sigma)\},$$

(ii) u_i^{**} is determined by the boundary conditions

$$u_i^{**} = 0 \quad \text{on} \quad x_3 = \pm l/2 \tag{2.3}$$

and traction components

$$T_\alpha^{**} = -\lambda n_\alpha, \quad T_3^{**} = 0$$

on the curved surface. It is readily verified using the Reciprocal Theorem, for example, that the energy ϵ^- associated with u_i^- is equal to the sum of the energies ϵ and ϵ^{**} associated with u_i and u_i^{**} respectively:

$$\epsilon^- = \epsilon + \epsilon^{**}. \tag{2.4}$$

Using

$$\begin{aligned} \epsilon &= \tfrac{1}{2}\bar{E}e^2A/l \\ \epsilon^- &= \tfrac{1}{2}E(1-\sigma)/\{(1+\sigma)(1-2\sigma)\}\cdot e^2A/l \end{aligned} \tag{2.5}$$

and the Principle of Minimum Complementary Energy, we obtain

$$(e^2A/l)\{E(1-\sigma)/(1+\sigma)(1-2\sigma) - E\} = 2\epsilon^{**} \le 2\int w(\bar{\tau}_{ij})\,dv \tag{2.6}$$

where $w(\bar{\tau}_{ij})$ denotes the strain energy density corresponding to a statistically admissible stress field $\bar{\tau}_{ij}$. Such a field is given by

$$\begin{aligned} \bar{\tau}_{\alpha\beta} &= -(\lambda e/l)\delta_{\alpha\beta}\Theta \\ \bar{\tau}_{\alpha 3} &= -(\lambda e/l)x_3\partial\Theta/\partial x_\alpha \\ \bar{\tau}_{33} &= (\lambda e/2l)x_3^2\nabla_1^2\Theta \end{aligned} \tag{2.7}$$

where $\Theta(x_1, x_2)$ satisfies

$$\Theta = 1, \quad \partial\Theta/\partial n = 0 \tag{2.8}$$

on the boundary $\mathscr{C}$ of the cross-section $\mathscr{D}$ of the cylinder, $\Theta \in C^2(\mathscr{D} U \mathscr{C})$ but is otherwise arbitrary. It follows from (2.5), (2.6) that

$$\{E(1-\sigma)/(1+\sigma)(1-2\sigma) - \bar{E}\} \le \{\lambda^2 l^4/(320EA)\}\int_{\mathscr{D}} \{(\nabla_1^2\Theta)^2 + (160/3)(1+2\sigma)l^{-2}(\nabla_1\Theta)^2 + 640(1-\sigma)l^{-4}\Theta^2\}\,dA. \tag{2.9}$$

Ideally, one would like to obtain the best possible (i.e. minimum) value of this integral for a general cross-section by solving the relevant Euler-Lagrange differential equation subject to the relevant boundary conditions, but as it is not possible to do this (except for special cross-sections such as a circle) the following approach is adopted as being the next best thing. Moreover, it gives rise to a simple explicit estimate.

Henceforward the domain of the cross-section $\mathscr{D}$ is supposed simply connected, and it is supposed that its boundary $\mathscr{C}$ (a Jordan curve) has continuous curvature. Transform $\mathscr{D}$ into a similar section $\bar{\mathscr{D}}$ of area π by a homogeneous uniform stretch

$$\bar{x}_\alpha = (\pi/A)^{1/2}x_\alpha.$$

Let $\bar{z} = \bar{x}_1 + i\bar{x}_2$ and transform $\bar{\mathscr{D}}$ onto the unit circle $\hat{\mathscr{D}}$ in the complex ζ plane by means of the conformal transformation $z = w(\zeta)$. It is known (e.g. [8]) that $|w'(\zeta)|$ is never zero or infinite in the closed domain when the boundary has the posited smoothness. The bound (2.9) then yields on suitable transplantation of the function Θ:

$$
\begin{aligned}
&E(1-\sigma)/\{(1+\sigma)(1-2\sigma)\} - \bar{E} \\
&\le \{\pi\lambda^2 l^4/320EA\} \int_{\bar{\mathscr{D}}} \{\bar{\nabla}_1^2\Theta)^2 \\
&+ (160/3)(1+2\sigma)\bar{l}^{-2}(\bar{\nabla}_1\Theta)^2 + 640(1-\sigma)\bar{l}^{-4}\Theta^2\}\, d\bar{A} \\
&\le \{\pi\lambda^2 l^4/(320EA)\} \int_{\tilde{\mathscr{D}}} \{J^{-1}(\tilde{\nabla}_1^2\Theta)^2 \qquad (2.10)\\
&+ (160/3)(1+2\sigma)\bar{l}^{-2}(\tilde{\nabla}_1\Theta)^2 + J640(1-\sigma)\bar{l}^{-4}\Theta^2\}\, d\tilde{A} \\
&\le \{M^2\pi\lambda^2 l^4/320EA)\} \int_{\tilde{\mathscr{D}}} \{\tilde{\nabla}_1^2\Theta)^2 \\
&+ (160/3)(1+2\sigma)\bar{l}^{-2}(\tilde{\nabla}_1\Theta)^2 + 640(1-\sigma)\bar{l}^{-4}\Theta^2\}\, d\tilde{A}
\end{aligned}
$$

where $\bar{l} = l/(A/\pi)^{1/2}$; $\bar{\nabla}_1$, $\tilde{\nabla}_1$ denote the gradient operators in the $\bar{z}$ and ζ planes respectively; $J = |d\bar{z}/d\zeta|^2$ and $M^2 = \max\{J, J^{-1}\}$. It is plain from the maximum modulus theorem that M is the greatest extension or contraction ratio occurring on the boundary resulting from the conformal transformation.

The functional (integral) in (2.10) is minimized by solving the appropriate Euler-Lagrange differential equation

$$\tilde{\nabla}_1^4\Theta - (160/3)(1+2\sigma)\bar{l}^{-2}\tilde{\nabla}_1^2\Theta + 640(1-\sigma)\bar{l}^{-4}\Theta = 0 \qquad (2.11)$$

in $\tilde{\mathscr{D}}$, subject to

$$\Theta = 1, \quad \frac{\partial\Theta}{\partial n} = 0 \quad \text{on} \quad \tilde{\mathscr{C}}.$$

The optimizing function Θ is plainly a function of the radial coordinate only and the optimum value is given in terms thereof by

$$-\int_{\mathscr{C}} \partial/\partial n(\nabla_1^2\Theta)\, ds.$$

One finds that the minimum value of the integral occurring in (2.10) (excluding the multiplicative factor) is

$$I_{\min} = 2\pi\bar{l}^{-3}\{\nu_1^3\nu_2 t(2) - \nu_1\nu_2^3 t(1)\}/\{\nu_1 t_1(1) - \nu_2 t(2)\} \qquad (2.12)$$

where

$$t(\alpha) = I_1(\nu_\alpha\bar{l}^{-1})/I_0(\nu_\alpha\bar{l}^{-1}).$$

I_0, I_1 denoting modified Bessel functions of orders zero and one respectively, and ν_1, ν_2 $(\nu_1 > \nu_2)$ are the positive roots of

$$\nu^4 - (160/3)(1+2\sigma)\nu^2 + 640(1-\sigma) = 0. \qquad (2.13)$$

It follows that

$$
\begin{aligned}
I_{\min} &\le (2/\bar{l}^3)\nu_1\nu_2(\nu_1 + \nu_2) \qquad (2.14)\\
&\le (2/\pi^{1/2})(A^{1/2}/l)^3 320\{(1+2\sigma)(1-\sigma)/3 + (1-\sigma)^{3/2}/\sqrt{10}\}^{1/2}
\end{aligned}
$$

where we have replaced the right-hand side of (2.12) by its asymptotic upper bound which it approaches rapidly with increasing argument (i.e. decreasing $l/A^{1/2}$) and where the equation (2.13) has been used. Since interest lies in the behaviour of the bound for small values of $l/A^{1/2}$, little is lost by replacing the upper bound (2.12) by the simpler explicit one (2.14)—particularly as the gain in transparency is great.

The resulting bound together with the nonnegativity of the difference state elastic energy ϵ^{**} [see (2.6)] gives rise to the following theorem:

Theorem 2: The modulus $\bar{E}$ satisfies

$$0 \leq 1 - \bar{E}/\{E(1-\sigma)/(1+\sigma)(1-2\sigma)\} \leq 2\pi^{1/2}M^2(l/A^{1/2})(1-2\sigma)^{-1} \sigma^2(1-\sigma^2)^{-1}\{(1+2\sigma)(1-\sigma)/3 + (1-\sigma)^{3/2}/\sqrt{10}\}^{1/2} \quad (2.15)$$

E, σ denoting Young's modulus and Poisson's ratio respectively, l, A the length and cross-sectional area respectively of the cylinder whose simply-connected cross-section is assumed to have a boundary with continuous curvature, and M is the greatest extension or contraction ratio on the boundary arising from the conformal transformation onto the interior of the unit circle of a cross-section of area π similar to that of the cylinder.

We note the following simple consequences of (2.15):

(a) $$\bar{E}/[E(1-\sigma)/\{(1+\sigma)(1-2\sigma)\}] \sim 1 \quad \text{as} \quad l/A^{1/2} \to 0.$$

In a discussion of an extension of Saint-Venant's principle Goodier [3], in effect, gives this result but without rigorous proof.

(b) $\bar{E}$ is asymptotically equivalent to the bulk modulus

$$\bar{E}/[E/\{3(1-2\sigma)\}] \sim 1$$

as $\sigma \to \frac{1}{2}$ (incompressibility) and $l/A^{1/2} \to 0$ in such a way that

$$(l/A^{1/2})(1-2\sigma)^{-1} \to 0.$$

(c) Considering a circular cylinder, consisting of foam rubber for which $\sigma = 0.39$, and which has diameter to length ratio 10:1, (2.15) yields $\bar{E}/E < 1.57$ (remembering $M = 1$). The value $\bar{E}/E \simeq 1.75$ is obtained from [5]. If this is taken to be the exact value the lower bound underestimates by about 10%.

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